15 Analytic functions

I start directly with the definition.

Definition 15.1. *Function $f\colon E \longrightarrow \mathbf{C}$ is called analytic in E if for any $z_0 \in E$ it can be represented by a convergent power series with non-zero radius of convergence.*

In the last section I showed that "analytic" implies "holomorphic." Here my first goal is to show that the converse is also true, and hence being analytic is a characterization of our main hero — holomorphic functions.

Theorem 15.2. *Let $f\colon E \longrightarrow \mathbf{C}$ be holomorphic. Then for each $z_0 \in E$ it can be represented as the power series $f(z) = \sum_{n\geq 0} c_n(z - z_0)^n$ with nonzero radius of convergence $R > 0$. Moreover, for each specific point $z_0 \in E$ the radius of convergence can be chosen as the minimum distance from point z_0 to the boundary of E.*

Proof. By the made assumptions, f is holomorphic in some ball $B(z_0, R)$. Let $g(z) = f(z + z_0)$, and g is holomorphic in the ball $B(0, R)$. For any $z \in B(0, R)$ and $|z| < r < R$ I can use the Cauchy's formula

$$g(z) = \frac{1}{2\pi \mathrm{i}} \int_{\partial B(0,r)} \frac{g(w)}{w - z} \mathrm{d}w.$$

Now I will use the fact that

$$\frac{1}{w-z} = \frac{1}{w}\frac{1}{1-\frac{z}{w}} = \frac{1}{w}\sum_{n\geq 0}\left(\frac{z}{w}\right)^n,$$

where the last series converges uniformly for all $w \in \partial B(0, r)$ by the Weierstrass M test, since $|z/w| < 1$ by construction. Since the series converges uniformly and all the terms are continuous, we are allowed to interchange the order of summation and integrations:

$$g(z) = \sum_{n\geq 0} z^n \frac{1}{2\pi \mathrm{i}} \int_{\partial B(0,r)} \frac{g(w)}{w^{n+1}} \mathrm{d}w,$$

and returning to the original function f I conclude that

$$f(z) = g(z - z_0) = \sum_{n\geq 0}(z - z_0)^n \frac{1}{2\pi \mathrm{i}} \int_{\partial B(z_0,r)} \frac{f(w)}{(w - z_0)^{n+1}} \mathrm{d}w = \sum_{n\geq 0} c_n(z - z_0)^n$$

as required. ∎

Therefore, in complex analysis the words "analytic" and "holomorphic" can be used interchangeably.

Note also that this theorem directly gives a recipe to determine R. For instance, if $f(z) = \tan z$ (recall the last section), and I need to compute power series around $z_0 = 0$ then the closest point at which tan stops being holomorphic is $a = \pm\pi/2$, then therefore the radius of convergence of the found power series is $R = \pi/2$.

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Now, using the power series representation, I can classify zeros of a holomorphic function. Recall that $\hat{z} \in E$ is a zero of $f\colon E \longrightarrow \mathbf{C}$ if $f(\hat{z}) = 0$. Since f is holomorphic at $\hat{z}$, I can represent it as a convergent power series

$$f(z) = c_1(z - \hat{z}) + c_2(z - \hat{z})^2 + \ldots,$$

note that $c_0 = 0$ because $f(\hat{z}) = 0$. Logically it is possible two have two possibility: 1) all the rest of c_n are zero, and in this case f is identically zero in some $B(\hat{z}, \epsilon)$, or 2) there is integer m such that $c_1 = \ldots = c_{m-1} = 0$ and $c_m \neq 0$. In this case I can write, by factoring $(z - \hat{z})^m$:

$$f(z) = (z - \hat{z})^m (c_m + c_{m+1}(z - \hat{z}) + c_{m+2}(z - \hat{z})^2 + \ldots) = (z - \hat{z})^m g(z),$$

where g is a holomorphic function (since it is represented by a convergent power series) and $g(\hat{z}) \neq 0$.

Now the constant m, exactly as in the case of a polynomial, is called a multiplicity of the *isolated* zero $\hat{z}$. If $m = 1$ then zero is called *simple*. The reasoning given above lead to the very important *identity principle*.

Theorem 15.3. *Let $f\colon E \longrightarrow \mathbf{C}$ be holomorphic and assume that $f(z_n) = 0$ where $(z_n)_{n=0}^{\infty}$ is a sequence of different numbers in E that converges in E. Then f is identically zero on E.*

Proof. The proof is topological in nature. Let me define two subsets of E:

$$X = \{z \in E\colon \text{there is } r \text{ such that } f(z) = 0 \text{ for all } z \in B(z, r)\},$$
$$Y = \{z \in E\colon \text{there is } r \text{ such that } f(z) \neq 0 \text{ for all } z \in B(z, r) \setminus \{z\}\}.$$

The key thing here is to realize that, due to our classification of zeros of f, $E = X \cup Y$ and $X \cap Y = \emptyset$. Indeed, if $z \in E$ then either $f(z) \neq 0$ or $f(z) = 0$. In the first case, due to continuity of f there will be a ball around z for which $f(z) \neq 0$. If $f(z) = 0$ then either f is identically zero (and $z \in X$) or it is isolated and hence $z \in Y$.

Now I need to show that both X and Y are open. Let $z \in X$. By definition $B(z, r) \subseteq X$ as well and therefore X is open. Now let $z \in Y$. Since the points in Y are either isolated zeros or non-zero, then $B(z, r) \subseteq Y$ for sufficiently small r, and hence Y is open. Since E is a union of two disjoint open sets and E is connected one of them must be empty. Let $\hat{z} \in E$ be the limit point of (z_n). By construction $\hat{z} \in X$, and hence Y is empty. Therefore, $X = E$ and f is identically zero. ∎

As an almost immediate corollary, I obtain

Corollary 15.4. *Let $f, g\colon E \longrightarrow \mathbf{C}$ be such that $f(z_n) = g(z_n)$ for a convergent in E sequence of distinct numbers (z_n). Then $f = g$ in E.*

Even more explicitly, if two holomorphic functions coincide on a small ball $B(z, r) \subseteq E$ then they must coincide on the whole domain E.

Finally, the power series representation allows proving the following *maximum principle* (also called *Maximum Modulus Theorem*).

Theorem 15.5. *Let $f\colon E \longrightarrow \mathbf{C}$ be holomorphic. Then $|f|$ cannot achieve a (local) maximum in E unless f is constant: if f is non-constant then for every $a \in E$ and $\delta > 0$ there is $z \in E$ with $|f(z)| > |f(a)|$ and $|z - a| < \delta$.*

Proof. Let f be non-constant and $a \in E$. Then I can write

$$f(z) = f(a) + c(z-a)^m + (z-a)^{m+1}h(z),$$

where m is a positive integer, c is nonzero complex number, and h is holomorphic. The idea of the proof is to find such z explicitly, for which $|f(a) + c(z-a)^m| = |f(a)| + |c(z-a)^m| > |f(a)|$ and the remaining term will be small enough to disregard.

Now to the exact details. Let $\alpha, \beta \in \mathbf{R}$ be such that $f(a) = |f(a)|e^{\mathrm{i}\alpha}$ and $c = |c|e^{\mathrm{i}\beta}$. Choose $\theta \in \mathbf{R}$ so that $\beta + m\theta = \alpha$ (and since $m \neq 0$ it is always possible). Now if $r > 0$ and $z = a + re^{\mathrm{i}\theta}$ I have

$$f(a) + c(z-a)^m = (|f(a)| + |c|r^m)e^{\mathrm{i}(\beta+m\theta)}.$$

That is $|f(a) + c(z-a)^m| = |f(a)| + |c|r^m$. Now, using the inverse triangle inequality,

$$|f(z)| = |f(a)+c(z-a)^m+(z-a)^{m+1}h(z)| \geq |f(a)+c(z-a)^m| - r^{m+1}|h(z)| \geq |f(a)|+|c|r^m - Mr^{m+1},$$

where I used the fact that being holomorphic for h implies that it is continuous around a and hence bounded by M for all $|z-a| \leq \rho$. Now I choose $\delta \in (0,\rho)$ such that $M\delta < |c|/2$. Then if $0 < r < \delta$,

$$|f(z)| \geq |f(a)| + \frac{|c|r^m}{2} > |f(a)|$$

as required. ■

Corollary 15.6. *Let $f\colon E \longrightarrow \mathbf{C}$ be holomorphic in E, E be bounded and extends continuously to ∂E. Then*

$$\sup_{z\in E} |f(z)| = \max_{z \in \partial E} |f(z)|.$$

Proof. If f is constant the statement is true. Assume f is not constant. Since $\overline{E} = E \cup \partial E$ is compact, $|f|$ reaches on it its maximum. By the proven maximum principle, this maximum cannot be in E, therefore, it is in ∂E. Since f is continuous everywhere in $\overline{E}$ the conclusion follows. ■